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Current-Loop Control in Switching Converters

Part 3: Waveform-Based Model Dynamics

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In this article, development of a unified model of current-mode control continues with the derivation of the dynamic equations for transfer functions of blocks in the current loop. The analysis begins with discrete-time waveform equations for inductor current and proceeds to derive s-domain expressions describing closed-loop operation of the converter.

To put this part in perspective, part 2 began with the general waveforms found in any of the three basic PWM-switch converter configurations. For each particular converter circuit type (*topology*), the inductor current can be derived from circuit analysis. The common-passive (buck), common-active (boost) or common-inductor configurations can all be controlled with a peak-current loop. From the inductor current triangle-wave, the basic discrete-time waveform and slope equations were derived in the previous part. These equations include the effects of incremental changes in the circuit variables but they do not tell us what the frequency-domain response of the current loop will be.

In this part, we transform the discrete-time equations from part 2 to the *z*-domain and pass quickly through it to the sampled *s*-domain. While I have attempted to explain some of the underlying theory for the reader who is unfamiliar with it, it nevertheless would be good to review or acquire an understanding of basic discrete-time *z* transforms, basic feedback-control theory, and also sampling theory.

Conversion of the waveform equations to the *z*-domain is straightforward and lets us derive a closed-loop transfer function of the peak-current controller. From the *z*-domain, it is also relatively easy to transform to the *s**, or sampled *s*-domain. A few of the less obvious subtleties are explained in some detail. Once we have the closed-loop transfer function, we proceed to derive the *s*-domain transfer function for the converter power-stage. Dynamics equations are derived for both the valley-current samples and for the average inductor current for each switching cycle. By deriving the transfer functions for the *average* current, a new refined model begins to emerge which shows a somewhat different response than the familiar valley-current model.

Closed-Loop Valley-Current Transfer Function

The valley-current transfer function results from converting the discrete-time waveform equations for inductor current to the *z*-domain by applying the time-shifting theorem:

$$Z\left\{f(t-k\cdot T_s)\right\}=F(z)\cdot z^k$$

where $z=e^{s\cdot T_s}$, a time advance of one cycle, T_s. The inductor current waveform equation of $i_l(k)$,

$$i_l(k)=-\left(\frac{m_D}{m_U}\right)\cdot i_l(k-1)+\left(1+\frac{m_D}{m_U}\right)\cdot i_i(k)$$

can be expressed in duty-ratio, D, instead of current waveform slopes:

$$i_l(k)=-\frac{D}{D'}\cdot i_l(k-1)+\frac{1}{D'}\cdot i_i(k)$$

When converted to the *z*-domain, it can be put in the form of the transfer function

$$T_{CV}(z)=\frac{i_l(z)}{i_i(z)}=\frac{1}{D'}\cdot\frac{z}{z+\left(\frac{D}{D'}\right)}=\frac{z}{D'\cdot z+D}.$$

Spending as little time as possible in the *z*-domain, the *z*-domain transfer functions are converted to the sampled (*) *s*-domain by substituting♣

$$z=e^{s\cdot T_s}.$$

This results in the sampled, closed-loop incremental transfer function for inductor current. For Ridley's sampled-loop model based on valley current,

$$T_{CV}(s)=\frac{i_l(s)}{i_i(s)}=T_{CV}^{*}(s)\cdot H_0(s)=\frac{1}{H_e(s)}\cdot\frac{1}{D'\cdot e^{s\cdot T_s}+D}=\frac{1}{\left(\frac{s}{\omega_s/2}\right)\cdot(\pi\cdot D')+H_e(s)}$$

where

$$H_e(s)=\frac{s\cdot T_s}{e^{s\cdot T_s}-1}$$

is Ridley's continuous-time feedback-path transfer function that accounts for loop sampling.

$$H_0(s)=\frac{1-e^{-s\cdot T_s}}{s\cdot T_s}$$

is the unitless (T_s-normalized) zero-order hold (ZOH) function that converts the sampled transfer function, $T_{CV}*(s)$, to a *piecewise-continuous*, quantized, or *stepped* function in time. The $1/T_s$ in $H_0(s)$ comes from the derivation of the transfer function where $1/T_s$ is in $d*(s)$ multiplied by $H_0(s)$;

$$d*(s)=\frac{1}{T_s}\cdot\sum_{n=-\infty}^{\infty}d(s-j\cdot n\cdot\omega_s).$$

Then $1/T_s$ is moved from $d*(s)$ into $H_0(s)$. This causes i_L* and $d*$ of the transfer-function ratio to have the units of i_L and d while the right-hand side also has the ratio of these units and $H_0(s)$ is unitless. The continuous $d(s)$ is in the first (n = 0) of the periodic Nyquist frequency bands. However, i_l and d in the sampled transfer function are not continuous and $d(s) \neq d*(s)\cdot H_0(s)$ but $d*(s)\cdot H_0(s)$ corresponds instead to the *stepped* cycle-averaged $d(t)$. Thus the i_l/d transfer function is a piecewise-continuous (stepped) function. For continuous recovery of $d(s)$, an ideal filter with cutoff frequency infinitesimally less than the Nyquist frequency ($\omega_S/2$) is required but is not causal and can only be approximated in electronic or software filters with a windowing filter such as a Hamming or Blackman filter.

This waveform quantization in the existing models nevertheless does not detract from their accuracy because the actual waveforms are stepped from circuit switching and are thus closely piecewise-continuous. Stepped waveforms are an accurate representation of the actual waveforms and no further improvement of their

♣ I am using the transform-aware convention that does not change the symbol of a quantity when it is transformed to a different domain (such as $x(t) \rightarrow x(s)$) but that the independent variable denotes the function corresponding to the domain. This convention is adopted because it is not uncommon to encounter in one analysis domains of *t*, *s*, *z*, *w*, and *s**, and use of different symbols (such as a capital letter for the *s*-domain, $X(s)$), becomes unwieldy.

accuracy is required. Tymerski noted that the step changes in i_L each cycle are the same as the output of a ZOH and that because the waveforms are stepped, the piecewise-continuous functions represent the actual waveforms.

In the time domain, $h_0(t)$ is a rectangular unit pulse of width T_s. One cycle of a function, $f(t)$, is turned on by the pulse, then turned off T_s later, thereby forming a gated step having a value during pulse k of $\bar{f}_k(t)$ with a constant value equal to that of the integrated ($1/s\cdot T_s$) impulse. A sampled function $f^*(t)$ convolved with rectangular pulse-train integrator $h_0(t)$ is converted to

$$\mathcal{L}^{-1}\{F^*(s)\cdot H_0(s)\} = f^*(t) * h_0(t) = \bar{f}(t)$$

where $\bar{f}(t)$ is *not* continuous but is the piecewise-continuous per-cycle-average stepped function of the continuous $f(t)$. The steps have high-frequency content containing harmonics in the bands outside the (n = 0) Nyquist band. In the frequency domain,

$$F^*(s) = \frac{1}{T_s}\cdot\sum_{n=-\infty}^{\infty} F(s - j\cdot n\cdot\omega_s) = F^*(s - j\cdot\omega_s).$$

$H_0(s)$ does not remove these steps and consequently does not remove all the frequencies in the harmonic ($n \neq 0$) bands.

In time, the stepped $\bar{f}(t)$ is shifted by $-T_s/2$ from that of $f(t)$, following from the phase relationship for H_0:

$$\angle H_0(j\omega) = -\frac{\pi}{2}\cdot\left(\frac{\omega}{\omega_s/2}\right) = (-90°)\cdot\left(\frac{f}{f_s/2}\right).$$

Consequently, the stepped $\bar{f}(t)$ lags behind $f(t)$ by $T_s/2$ in time with a phase shift of $\omega\cdot(-T_s/2)$ in radians. This is accounted for in frequency-domain loop modeling by use of $H_0(s)$ or its associated $H_e(s)$.

The $i_L(t)$ of the "continuous-time" model is also piecewise-continuous. Ridley and Tymerski recognized that the zero-order-hold stepped function of incremental sensed inductor current is not exact. By subtracting the steady-state waveform from the perturbed waveform, the cyclic change in i_L follows a ramp of current with $\pm m_U$ slope. Then incremental i_L is truly continuous, though it is not continuously differentiable because i_L is a piecewise-linear function.

Using the two-point fit♣ of Tymerski and Ridley,

$$H_e(s) \cong \left(\frac{s}{\omega_s/2}\right)^2 - \frac{\pi}{2}\cdot\left(\frac{s}{\omega_s/2}\right) + 1.$$

The linear pole terms add to the result in

♣ This is often referred to as a *modified* Padé rational-function approximation. Tymerski explains in some detail that departure from the Padé approximation is made to achieve an approximation that is more accurate in the Nyquist interval, which is "where the action is" in sampled systems. The two-point fit is easily derived by solving for coefficients at s = 0 and $\omega_s/2$.

$$T_{CV}(s) \cong \frac{1}{\left(\frac{s}{\omega_s/2}\right)^2 + \pi \cdot (\tfrac{1}{2} - D) \cdot \left(\frac{s}{\omega_s/2}\right) + 1}\ .$$

From the linear pole coefficient it is evident that $D < ½$ for a LHP pole-pair. The damping is

$$\zeta = \frac{1}{2 \cdot Q} = \frac{\pi}{2} \cdot (\tfrac{1}{2} - D)\,.$$

Thus the damping varies with D. Some values of pole-pair locations and damping are given in the following table with the following parameters: ϕ = pole angle in degrees; M_p = fractional overshoot of a unit step for a quadratic response; N_s = number of cycles of ringing observable at 8-bit resolution, comparable to an oscilloscope screen.

$\phi = \cos^{-1}(\zeta)$; $M_p = e^{-\pi/\tan\phi}$; $N_s \approx (0.11)\cdot\tan\phi\log_2(1/M_s) = (0.11)\cdot\tan\phi 8 \approx 0.88\cdot\tan\phi$

Table. Pole-pair locations for $T_{CV}(s)$.

D	ζ	ϕ, °	M_p, %	N_s	
0	$\pi/4 \approx 0.785$	38.2	1.86	0.7	max ζ
0.05	$\sqrt{2}/2 \approx 0.707$	45	4.32	0.9	MFA
0.1817	½ = 0.500	60	16.30	1.5	
1/3 ≈ 0.333	$\pi/8 \approx 0.393$	66.88	26.15	2.1	
¼ = 0.25	$\pi/12 \approx 0.262$	74.82	42.65	3.3	

The minimum value of pole angle is more than that of a maximally-flat envelope delay (MFED) response (30°) because of the discrete-time nature of the loop. It is evident that even at half the value of D for which T_{CV} becomes undamped (D = ¼), the fractional overshoot to a load-current step is considerable, about 43%.

Valley And Average Currents Are Dynamically Different

A difference between the sampled-loop model and the lf-avg and unified models is the inductor current quantities. The unified-model inductor current as it applies to the PWM block is based on the discrete-time average $i_L(k)$ over a cycle, or $\bar{i}_L(k)$. Tymerski's early unified state-space model is based on average current. The sampled-loop model is based on the minimum (valley) current values of each cycle. Consequently, equating the sampled-loop i_l / i_i with the feedback formula results in a different PWM transfer function, F_m, because i_l of the sampled-loop transfer function is valley current, not average current.

The discrete-time per-cycle average inductor current, $\bar{i}_L(k)$, is found by averaging the on- and off-time current averages, where the $i_L(k)$ are the valley points ending each cycle;

$$\bar{i}_L(k) = \tfrac{1}{2}\cdot(i_I(k) + i_L(k-1))\cdot D + \tfrac{1}{2}\cdot(i_I(k) + i_L(k))\cdot D' = \tfrac{1}{2}\cdot i_I(k) + \tfrac{1}{2}\cdot(i_L(k-1)\cdot D + i_L(k)\cdot D')\,.$$

Under steady-state operation, the cycle-minimum or valley current values are $i_L(k-1) = i_L(k) = i_{LV}$ and

$$\bar{i}_L = \tfrac{1}{2}\cdot(i_I + i_{LV})$$

The incremental valley and average currents for cycle k, respectively $i_l(k)$ and $\bar{i}_l(k)$, can be produced by differentiating the total-variable relationship,

$$\bar{i}_L(k) = \tfrac{1}{2}\cdot\left(i_L(k) + i_I(k)\right)$$

From the current waveform plots, $i_L(k)$ is the final value of i_L for cycle k, and $\bar{i}_L(k)$ ends at the peak i_L within (the next) cycle (k + 1). Consequently, $\bar{i}_L(k)$ is calculated as the average value of $i_L(k)$ and $i_I(k)$ though it is also the steady-state average value of i_L during the up-slope of cycle (k + 1) until just before i_I changes to $i_I(k + 1)$.

The above average is calculated from the down-slope segment endpoints of cycle k. With $\Delta i_I = 0$, $\bar{i}_L(k)$ varies by half that of the valley-current sample, $i_L(k)$. Both the final value of i_L and the peak value, i_I, are of the same cycle. Solving for i_L as a function of $\bar{i}_L$,

$$i_L(k) = 2\cdot\bar{i}_L(k) - i_I(k)\,.$$

Then, taking the differential of $i_L(k)$, the incremental relationship is

$$i_l(k) = 2\cdot\bar{i}_l(k) - i_i(k)\ .$$

Going back to the waveform equations and expressing them in δ instead of slopes, they are

$$i_l(k) = -\left(\frac{\delta}{\delta'}\right)\cdot i_l(k-1) + \frac{1}{\delta'}\cdot i_i(k)$$

$$d(k) = \frac{i_l(k) - i_l(k-1)}{(v_{OFF}/L)\cdot T_s} = \frac{i_l(k) - i_l(k-1)}{\Delta i_{L0}}$$

where

$$\Delta i_{L0} = \frac{v_{OFF}\cdot T_s}{L}\ ;\ \Delta I_{L0} = \frac{V_{off}\cdot T_s}{L}\,;\,\Delta i_{l0} = \frac{v_{off}\cdot T_s}{L}$$

In steady-state operation, $\delta = D$ and $i_l(\bar{i}_l)$ can be substituted into the general incremental waveform equation

$$i_l(k) = -\frac{D}{D'}\cdot i_l(k-1) + \frac{1}{D'}\cdot i_i(k)\,.$$

Substituting the above average current expression, the current equation then becomes

$$2\cdot\bar{i}_l(k) - i_i(k) = -\frac{D}{D'}\cdot[2\cdot\bar{i}_l(k-1) - i_i(k-1)] + \frac{1}{D'}\cdot i_i(k)\,.$$

Collecting terms and solving,

$$\bar{i}_l(k) = -\frac{D}{D'}\cdot\bar{i}_l(k-1) + \frac{1}{2}\cdot\left[\left(1+\frac{1}{D'}\right)\cdot i_i(k) + \frac{D}{D'}\cdot i_i(k-1)\right] = -\frac{D}{D'}\cdot\left(\bar{i}_l(k-1) + \tfrac{1}{2}\cdot[i_i(k) - i_i(k-1)]\right) + \frac{1}{D'}\cdot i_i(k).$$

The rightmost expression has the form of the valley-current expression but with the additional i_i difference term added to $\bar{i}_l(k-1)$. For a constant rate of change, $i_i(k-1) = i_i(k)$; then $\bar{i}_l(k) = i_l(k)$. Only for changing i_i are they different.

Taking the z transform,

$$\bar{i}_l(z) = -\frac{D}{D'}\cdot\bar{i}_l(z)\cdot z^{-1} + \frac{1}{2}\cdot\frac{1}{D'}\cdot\left(D'+1+D\cdot z^{-1}\right)\cdot i_i(z)$$

Then solving for the closed-loop current transfer function,

$$T_C(z) = \frac{\bar{i}_l(z)}{i_i(z)} = \frac{1/2}{D'}\cdot\frac{(D'+1)\cdot z + D}{z+\dfrac{D}{D'}} = \frac{\tfrac{1}{2}\cdot\left[(D'+1)\cdot z + D\right]}{D'\cdot z + D} = \tfrac{1}{2}\cdot\left(1+T_{CV}(z)\right)$$

This result could more readily have been obtained by noting that not only is $\bar{i}_l$ a function of i_l, it is also a function of i_i, or

$$\bar{i}_l = \tfrac{1}{2}\cdot\left(i_l(i_i) + i_i\right) = \tfrac{1}{2}\cdot\left(T_{CV}\cdot i_i + i_i\right) = \tfrac{1}{2}\cdot\left(1+T_{CV}\right)\cdot i_i$$

After steady-state substitution for $i_l(\bar{i}_l)$, $d(\bar{i}_l)$ is

$$d(k) = \frac{2\cdot(\bar{i}_l(k) - \bar{i}_l(k-1)) - (i_i(k) - i_i(k-1))}{(v_{OFF}/L)\cdot T_s} = \frac{2\cdot(\bar{i}_l(k) - \bar{i}_l(k-1)) - (i_i(k) - i_i(k-1))}{\Delta i_{L0}}$$

Converter Transfer Function

The incremental valley-current waveform equation for $d(k)$ was derived as

$$d(k) = \frac{i_l(k) - i_l(k-1)}{(v_{OFF}/L)\cdot T_s} = \frac{i_l(k) - i_l(k-1)}{\Delta i_{L0}}.$$

Solving for the steady-state i_l equation, it converts to the z-domain as

$$i_l(z) = z^{-1}\cdot i_l(z) + \frac{V_{off}\cdot T_s}{L}\cdot d(z) = z^{-1}\cdot i_l(z) + \Delta I_{L0}\cdot d(z).$$

As a transfer-function,

$$G_{idV}(z) = \frac{i_l(z)}{d(z)} = \left(\frac{z}{z-1}\right)\cdot\Delta I_{L0}.$$

This can be transformed directly to the s-domain in that $z/(z-1)$ transforms as $1/s$. It can alternatively be transformed to the sampled s-domain by substitution of $z = e^{s\cdot T_s}$ to result in

$$\frac{i_l^*(s)}{d^*(s)} = \left(\frac{e^{s\cdot T_s}}{e^{s\cdot T_s}-1}\right)\cdot\left(\frac{V_{off}\cdot T_s}{L}\right) = \left(\frac{1}{1-e^{-s\cdot T_s}}\right)\cdot \Delta I_{L0}\,.$$

Then the sampled $i_l^*(s)$ can be recovered in stepped form by a zero-order hold, $H_0(s)$;

$$G_{idV} = \frac{i_l(s)}{d(s)} = \frac{i_l^*(s)}{d^*(s)}\cdot H_0(s) = \left(\frac{1}{1-e^{-s\cdot T_s}}\right)\cdot\frac{V_{off}\cdot T_s}{L}\cdot\left(\frac{1-e^{-s\cdot T_s}}{s\cdot T_s}\right) = \frac{\Delta I_{L0}}{s\cdot T_s} = \frac{V_{off}}{s\cdot L}\,.$$

The resulting $i_l\,/d$ transfer function is independent of converter configuration in that it is derived from the closed-loop waveforms. It is found in the Ridley model.

Because i_l is the incremental valley current and not $\bar{i}_l$, for the refined unified model the average and not valley current is used. By substituting

$$i_l(k) = 2\cdot\bar{i}_l(k) - i_i(k)$$

into the incremental current equation,

$$i_l(k) = i_l(k-1) + \frac{V_{off}\cdot T_s}{L}\cdot d(k) = i_l(k-1) + \Delta I_{L0}\cdot d(k)$$

and setting i_i to zero, transforming to the z-domain, and simplifying,

$$G_{id} = \frac{\bar{i}_l(z)}{d(z)} = \frac{1}{2}\cdot\frac{V_{off}\cdot T_s}{L}\cdot\frac{z}{z-1} = \frac{1}{2}\cdot\Delta I_{L0}\cdot\frac{z}{z-1} = \tfrac{1}{2}\cdot G_{idV} = \frac{1}{2}\cdot\frac{V_{off}}{s\cdot L}\,.$$

Converted to the s-domain,

$$G_{id}(s) = \frac{\bar{i}_l(s)}{d(s)} = \frac{1}{2}\cdot\frac{i_l(s)}{d(s)} = \frac{\Delta I_{L0}}{2\cdot s\cdot T_s}\,.$$

and is the converter transfer function, G_{id}. It differs from the valley-current G_{idV} by the ½ factor. From either G_{id} or G_{idV} it is apparent (as Sheehan notes) that the converter is basically an inductor integrating its applied input voltage as an output current.

The closed-loop transfer function, $T_{CV}(s)$, as derived from the discrete-time waveforms is independent of any particular converter topology given that its inductor current waveform is the current waveform from which the transfer function was derived. This also applies to G_{id} and G_{idV}. Consequently, with i_{Ce} sampling, $T_{CV}(s)$ can be equated to the general circuit feedback loop equation.

Closure

With some mathematical effort, the discrete-time waveform equations have led to transfer functions in the s-domain. The discrete-time equations for average inductor current were developed and converted to the s-domain. The average-current transfer function has a different dynamic response than the valley-current transfer function. In the next part, clarifications of existing models are presented.

About The Author

Dennis Feucht has been involved in power electronics for 25 years, designing motor-drives and power converters. He has an instrument background from Tektronix, where he designed test and measurement equipment and did research in Tek Labs. He has lately been doing current-loop converter modeling and converter optimization.

For more on current-mode control methods, see the How2Power Design Guide, select the Advanced Search option, go to Search by Design Guide Category, and select “Control Methods” in the Design Area category.